

$f(R)$ cosmology by Noether's symmetry

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A general approach to find out exact cosmological solutions in $f(R)$ -gravity is discussed. Instead of taking into account phenomenological models, we assume, as a physical criterium, the existence of Noether symmetries in the cosmological $f(R)$ Lagrangian. As a result, the presence of such symmetries selects viable models and allow to solve the equations of motion. We discuss also the case in which no Noether charge is present but general criteria can be used to achieve solutions.

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I. INTRODUCTION

The recent issue to investigate alternative theories of gravity comes out from Cosmology, Quantum Field Theory and Mach's Principle. The initial singularity, the flatness and horizon problems [1] point out that Standard Cosmological Model [2], based on General Relativity (GR) and Particle Standard Model, fails in describing the Universe at extreme regimes. Besides, GR does not work as a fundamental theory capable of giving a quantum description of spacetime. Due to these reasons and to the lack of a definitive Quantum Gravity theory, alternative theories of gravitation have been pursued in order to attempt, at least, a semi-classical approach to quantization. In particular, *Extended Theories of Gravity* (ETGs) face the problem of gravitational interaction correcting and enlarging the Einstein theory.

The general paradigm consists in adding, into the effective action, physically motivated higher-order curvature invariants and non-minimally coupled scalar fields [3, 4].

The interest of such an approach in early epoch cosmology is due to the fact that ETGs can “naturally” reproduce inflationary behaviors able to overcome the shortcomings of the Standard Cosmological Model and seems also capable of matching with several observations.

From another viewpoint, the Mach Principle gives further motivations to modify GR stating that the local inertial frame is determined by the average motion of distant astronomical objects [5]. As a consequence, the gravitational coupling can be scale-dependent. This means that the concept of inertia and the Equivalence Principle have to be revised since there is no *a priori* reason to restrict the gravitational Lagrangian to a linear function of the Ricci scalar R , minimally coupled with matter [6–11].

Very recently, ETGs are playing an interesting role to describe today's observed Universe. In fact, the impressive amount of good quality data of last decade seems to shed new light into the effective picture of the Universe. Type Ia Supernovae (SNeIa) [12], anisotropies in the CMBR [13], and matter power spectrum derived from wide and deep galaxy surveys [14] represent the strongest evidences for a radical revision of the Cosmological Standard Model also at recent epochs.

Specifically, the *Concordance Λ CDM Model* is showing that baryons contribute only for $\sim 4\%$ to the total matter-energy budget, while the *cold dark matter* (CDM) represents the bulk of the clustered large scale structures ($\sim 25\%$) and the cosmological constant Λ plays the role of the so called “dark energy” ($\sim 70\%$) [15].

Although being the best fit to a wide range of data [16], the Λ CDM model is affected by strong theoretical shortcomings [17] that have motivated the search for alternative models [18, 19].

Dark energy models mainly rely on the implicit assumption that Einstein's GR is the correct theory of gravity indeed. Nevertheless, its validity on large astrophysical and cosmological scales has never been tested but only *assumed* [20], and it is therefore conceivable that both cosmic speed up and missing matter are nothing else but

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signals of a breakdown of GR. In this sense, GR could fail in giving self-consistent pictures both at ultraviolet scales (early universe) and at infrared scales (late universe).

Following this line of thinking, the “minimal” choice could be to take into account generic functions $f(R)$ of the Ricci scalar R . However, such an approach can be encompassed in the ETGs being the minimal extension of GR. The task for this extended theories should be to match the data under the “economic” requirement that no exotic dark ingredients have to be added, unless these are going to be found with fundamental experiments [21]. This is the underlying philosophy of what are referred to as $f(R)$ -gravity (see [19, 22, 23] and references therein).

Although higher order gravity theories have received much attention in cosmology, since they are naturally able to give rise to the accelerating expansion (both in the late and in the early universe [24]), it is possible to demonstrate that $f(R)$ theories can also play a major role at astrophysical scales. In fact, modifying the gravity Lagrangian affects the gravitational potential in the low energy limit. Provided that the modified potential reduces to the Newtonian one on the Solar System scale, this implication could represent an intriguing opportunity rather than a shortcoming for $f(R)$ theories. In fact, a corrected gravitational potential could offer the possibility to fit galaxy rotation curves without the need of huge amounts of dark matter [25–31]. In addition, it is possible to work out a formal analogy between the corrections to the Newtonian potential and the usually adopted galaxy halo models which allow to reproduce dynamics and observation *without* dark matter [27].

However, extending the gravitational Lagrangian could give rise to several problems. These theories could have instabilities [32, 33] and ghost-like behaviors [34–36], and they have to be matched with the low energy limit experiments which fairly test GR. Besides, these theories should also be compatible with early universe tests such as the formation of CMBR anisotropies, Big Bang Nucleosynthesis [37], and Baryogenesis [38, 39].

Actually, the debate concerning the weak field limit of $f(R)$ -gravity is far to be definitive. In the last few years, several authors have dealt with this matter with contrasting conclusions, in particular with respect to the Parameterized Post Newtonian (PPN) limit [40, 42].

In summary, it seems that the paradigm to adopt $f(R)$ -gravity leads to interesting results at cosmological, galactic and Solar System scales but, up to now, no definite physical criterion has been found to *select* the final $f(R)$ theory (or class of theories) capable of matching the data at all scales. Interesting results have been achieved in this line of thinking [21, 43–46] but the approaches are all phenomenological and are not based on some fundamental principle as the conservation or the invariance of some quantity or some intrinsic symmetry of the theory. Furthermore, as it was shown in [32], in alternative theories of gravity, it is important to understand the background before exploring other bounds, such as anisotropies in the CMBR. For this goal it is essential to try to find exact analytical solutions for the $f(R)$ theories, and, only after this, study more in detail the possible evolutions compatible with our data (e.g. solar system and CMBR bounds).

In some sense, the situation is similar to that of dark matter: we know very well its effect at large astrophysical scales but no final evidence of its existence has been found, up to now, at fundamental level. In the case of $f(R)$ -gravity, we know that the paradigm is working: in principle, the missing matter and accelerated cosmic behavior can be addressed taking into account gravity (in some extended version), baryons and radiation but we do not know a specific criterion to select the final, comprehensive theory.

In this paper, we want to address the following issues: *i*) Is there some general principle capable of *selecting* physically motivated $f(R)$ models? *ii*) Can conserved quantities or symmetries be found in relation to specific $f(R)$ theories? *iii*) Can such quantities, if existing, give rise to viable cosmological models?

In this paper, following the so called *Noether Symmetry Approach* (see [7, 47, 48]), we want to seek for viable $f(R)$ cosmological models. As we will see, the method is twofold: from one side, the existence of symmetries allows to solve exactly the dynamics; from the other side, the Noether *charge* can always be related to some observable quantity.

The layout of the paper is the following. In Sec.II, we sketch the dynamics of $f(R)$ gravity in the metric approach and derive the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological equations. Sec.III is devoted to the general discussion of the Noether Symmetry Approach by which it is possible to find out conserved quantities and then symmetries which allow to exactly solve a dynamical system. In Sec.IV, we apply the method to the $f(R)$ cosmology. In Sec.V, we give a detailed summary of the exact solutions discussing them in presence or in absence of the Noether charge. Sec.VI is devoted to the discussion and the conclusions.

II. $f(R)$ GRAVITY AND COSMOLOGY

The action

$$S = \int d^4x \sqrt{-g} f(R) + S_m, \quad (1)$$

describes a theory of gravity where $f(R)$ is a generic function of the Ricci scalar R . GR is recovered in the particular case $f(R) = -R/16\pi G$, and S_m is the action for a perfect fluid minimally coupled with gravity¹.

This action, in general, leads to 4th order differential equations for the metric since the field equations are

$$f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu} \square f_R = -\frac{1}{2} T_{\mu\nu}^m, \quad (2)$$

where a subscript R denotes differentiation with respect to R and $T_{\mu\nu}^m$ is the matter fluid stress-energy tensor.

Defining a *curvature stress-energy tensor* as

$$T_{\mu\nu}^{curv} = \frac{1}{f_R(R)} \left\{ \frac{1}{2} g_{\mu\nu} [f(R) - R f_R(R)] + f_R(R)^{\alpha\beta} (g_{\alpha\mu} g_{\beta\nu} - g_{\mu\nu} g_{\alpha\beta}) \right\}, \quad (3)$$

Eqs.(2) can be recast in the Einstein-like form:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}^{curv} + T_{\mu\nu}^m / f_R(R) \quad (4)$$

where matter non-minimally couples to geometry through the term $1/f_R(R)$. It is known that these theories can be mapped to a scalar-tensor theory. However, there are two points which should be noticed. First, the two theories might have different quantum descriptions, as they only coincide on the classical solutions. Furthermore, the two theories are classically equivalent if the Brans-Dicke parameter (ω_{BD}) exactly vanishes and if the scalar field possesses a suitable potential. This fact is related to the second point: in the literature, the Brans-Dicke field is commonly taken as a light scalar field for which the local gravity constraint fixes the Brans-Dicke parameter to be greater than 40000. This bound is usually considered when studying Brans-Dicke theories. However, for the $f(R)$ theories, since $\omega_{BD} = 0$, this is not the case, and the presence of a non-negligible potential is essential in order to give an explicit mass to the gravitational scalar degree of freedom. Once one has the solution $H(t)$ (and consequently $R(t)$) for a given $f(R)$, the scalar field is defined as $\Phi(t) = -f_R(t)$, and its potential is $U(\Phi(t)) = R(t) f_R(t) - f(R(t))$. An example showing this link between scalar-tensor theories and $f(R)$ gravity is given in the appendix for one solution which will be found explicitly later on.

In order to derive the cosmological equations in a FLRW metric, one can define a canonical Lagrangian $\mathcal{L} = \mathcal{L}(a, \dot{a}, R, \dot{R})$, where $\mathcal{Q} = \{a, R\}$ is the configuration space and $\mathcal{TQ} = \{a, \dot{a}, R, \dot{R}\}$ is the related tangent bundle on which \mathcal{L} is defined. The variable $a(t)$ and $R(t)$ are the scale factor and the Ricci scalar in the FLRW metric, respectively. One can use the method of the Lagrange multipliers to set R as a constraint of the dynamics. Selecting the suitable Lagrange multiplier and integrating by parts, the Lagrangian \mathcal{L} becomes canonical. In our case, we have

$$S = 2\pi^2 \int dt a^3 \left\{ f(R) - \lambda \left[R + 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} \right) \right] - \frac{\rho_{m0}}{a^3} - \frac{\rho_{r0}}{a^4} \right\}, \quad (5)$$

where a is the scale factor scaled with respect to today's value (so that $a = \tilde{a}/\tilde{a}_0$ and $a(t_0) = 1$); ρ_{m0} and ρ_{r0} represent the standard amounts of dust and radiation fluids as, for example, measured today; finally $\kappa = k/\tilde{a}_0^2$, where $k = 0, \pm 1$. This choice for a , makes it dimensionless, and it also implies that $[\kappa] = [R] = M^2$, whereas $[f] = [\rho_{r0}] = M^4$. It is straightforward to show that, for $f(R) = -R/16\pi G - \rho_{\Lambda 0}$, one obtains the usual Friedmann equations.

The variation with respect to R of the action gives $\lambda = f_R$. Therefore the previous action can be rewritten as

$$S = 2\pi^2 \int dt a^3 \left\{ f - f_R \left[R + 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} \right) \right] - \frac{\rho_{m0}}{a^3} - \frac{\rho_{r0}}{a^4} \right\}, \quad (6)$$

and then, integrating by parts, the point-like FLRW Lagrangian is

$$\mathcal{L} = a^3 (f - f_R R) + 6 a^2 f_{RR} \dot{R} \dot{a} + 6 f_R a \dot{a}^2 - 6\kappa f_R a - \rho_{m0} - \rho_{r0}/a, \quad (7)$$

which is a canonical function of two coupled fields, R and a , both depending on time t . The total energy $E_{\mathcal{L}}$, corresponding to the 0,0-Einstein equation, is

$$E_{\mathcal{L}} = 6 f_{RR} a^2 \dot{a} \dot{R} + 6 f_R a \dot{a}^2 - a^3 (f - f_R R) + 6\kappa f_R a + \rho_{m0} + \frac{\rho_{r0}}{a} = 0. \quad (8)$$

¹ We are using the following conventions, $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, and $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$, $c = \hbar = 1$.

As we shall see later, it is convenient to look for parametric solutions in the form $[H(a), f(R(a))]$, so that $f_R = f'/R'$, where a prime denotes differentiation with respect to the time-parameter a . We also have that, if $R \neq \text{constant}$, $f_{RR} \dot{R} = df_R/dt = a H f'_R = a H [f''/R' - f' R''/R'^2]$, so that the Friedmann equation can be rewritten as

$$f - 6a \left(\frac{f''}{R'} - \frac{f' R''}{R'^2} \right) H^2 - \frac{6f' H^2}{R'} - \left(\frac{6\kappa}{a^2} + R \right) \frac{f'}{R'} = \frac{\rho_{0m}}{a^3} + \frac{\rho_{0r}}{a^4}. \quad (9)$$

The equations of motion for a and R are respectively

$$f_{RR} \left[R + 6H^2 + 6\frac{\ddot{a}}{a} + 6\frac{\kappa}{a^2} \right] = 0 \quad (10)$$

$$6f_{RRR} \dot{R}^2 + 6f_{RR} \ddot{R} + 6f_R H^2 + 12f_R \frac{\ddot{a}}{a} = 3(f - f_R R) - 12f_{RR} H \dot{R} - 6f_R \frac{\kappa}{a^2} + \frac{\rho_{r0}}{a^4}, \quad (11)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. Considering R and a as the variables, we have, for consistency (excluding the case $f_{RR} = 0$), that R coincides with the definition of the Ricci scalar in the FLRW metric. Geometrically, this is the Euler constraint of the dynamics. Using (10), only one of the equations (8), and (11) is independent because of the Bianchi identities, as these equations correspond to the first and second modified Einstein equations, and matter is conserved. Equivalently, after multiplying equation (11) by $a^2 \dot{a}$, and using (10), one can integrate (11) to find (8). Furthermore, as we will show below, constraints on the form of the function $f(R)$ and, consequently, solutions of the system (8), (10) can be achieved by asking for the existence of Noether symmetries. Such solutions will also solve equation (11) automatically. On the other hand, the existence of the Noether symmetries guarantees the reduction of dynamics and the eventual solvability of the system.

III. THE NOETHER SYMMETRY APPROACH

Solutions for the dynamics given by (7) can be achieved by selecting cyclic variables related to some Noether symmetry. In principle, this approach allows to select $f(R)$ -gravity models compatible with the symmetry so it can be seen as a physical criterion since the conserved quantities are a sort of Noether charges. Therefore such a criterion might be to look for those $f(R)$ which have *cosmological* Noether charge. Although this criterion somehow “breaks” Lorentz-invariance because we need the FLRW background to formulate it, however Lorentz-invariance is evidently broken in our universe by the presence of the CBMR radiation which, by itself, fixes a preferred reference frame.

In general, the Noether Theorem states that conserved quantities are related to the existence of cyclic variables into dynamics [49–51].

Let $\mathcal{L}(q^i, \dot{q}^i)$ be a canonical, non-degenerate point-like Lagrangian where

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0; \quad \det H_{ij} \stackrel{\text{def}}{=} \det \left\| \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j} \right\| \neq 0, \quad (12)$$

with H_{ij} the Hessian matrix related to \mathcal{L} and a dot denotes differentiation with respect to the affine parameter λ . The dot indicates derivatives with respect to the affine parameter λ which, in our case, corresponds to the cosmic time t . In analytical mechanics, \mathcal{L} is of the form

$$\mathcal{L} = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q}), \quad (13)$$

where T and V are the “kinetic” and “potential energy” respectively. T is a positive definite quadratic form in $\dot{\mathbf{q}}$. The energy function associated with \mathcal{L} is

$$E_{\mathcal{L}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \dot{q}^i - \mathcal{L}, \quad (14)$$

which is the total energy $T + V$. In any case, $E_{\mathcal{L}}$ is a constant of motion. Since our cosmological problem has a finite number of degrees of freedom, we are going to consider only point-transformations. Any invertible transformation of the “generalized positions” $Q^i = Q^i(\mathbf{q})$ induces a transformation of the “generalized velocities” such that

$$\dot{Q}^i(\mathbf{q}) = \frac{\partial Q^i}{\partial q^j} \dot{q}^j; \quad (15)$$

the matrix $\mathcal{J} = \|\partial Q^i / \partial q^j\|$ is the Jacobian of the transformation on the positions, and it is assumed to be nonzero. The Jacobian $\tilde{\mathcal{J}}$ of the induced transformation is easily derived and $\mathcal{J} \neq 0 \rightarrow \tilde{\mathcal{J}} \neq 0$. In general, this condition is not satisfied in the whole space but only in the neighbor of a point. It is a local transformation.

A point transformation $Q^i = Q^i(\mathbf{q})$ can depend on one (or more than one) parameter. As starting point, we can assume that a point transformation depends on a parameter ε , i.e. $Q^i = Q^i(\mathbf{q}, \varepsilon)$, and that it gives rise to a one-parameter Lie group. For infinitesimal values of ε , the transformation is then generated by a vector field: for instance, $\partial/\partial x$ is a translation along the x axis, $x(\partial/\partial y) - y(\partial/\partial x)$ is a rotation around the z axis and so on. The induced transformation (15) is then represented by

$$\mathbf{X} = \alpha^i(\mathbf{q}) \frac{\partial}{\partial q^i} + \left(\frac{d}{d\lambda} \alpha^i(\mathbf{q}) \right) \frac{\partial}{\partial \dot{q}^i} . \quad (16)$$

\mathbf{X} is called the “complete lift” of \mathbf{X} [51]. A function $F(\mathbf{q}, \dot{\mathbf{q}})$ is invariant under the transformation \mathbf{X} if

$$L_{\mathbf{X}} F \stackrel{\text{def}}{=} \alpha^i(\mathbf{q}) \frac{\partial F}{\partial q^i} + \left(\frac{d}{d\lambda} \alpha^i(\mathbf{q}) \right) \frac{\partial F}{\partial \dot{q}^i} = 0 , \quad (17)$$

where $L_{\mathbf{X}} F$ is the Lie derivative of F . Specifically, if $L_{\mathbf{X}} \mathcal{L} = 0$, \mathbf{X} is a *symmetry* for the dynamics derived by \mathcal{L} . As we shall see later on, we will look for a sufficient condition on the form of $f(R)$ in our Lagrangian, which allows $L_{\mathbf{X}} \mathcal{L} = 0$ to vanish.

Let us consider now a Lagrangian \mathcal{L} and its Euler-Lagrange equations

$$\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{q}^j} - \frac{\partial \mathcal{L}}{\partial q^j} = 0 . \quad (18)$$

Let us consider also the vector field (16). Contracting (18) with the α^i 's gives

$$\alpha^j \left(\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{q}^j} - \frac{\partial \mathcal{L}}{\partial q^j} \right) = 0 . \quad (19)$$

Being

$$\alpha^j \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{q}^j} = \frac{d}{d\lambda} \left(\alpha^j \frac{\partial \mathcal{L}}{\partial \dot{q}^j} \right) - \left(\frac{d\alpha^j}{d\lambda} \right) \frac{\partial \mathcal{L}}{\partial \dot{q}^j} , \quad (20)$$

from (19), we obtain

$$\frac{d}{d\lambda} \left(\alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = L_{\mathbf{X}} \mathcal{L} . \quad (21)$$

The immediate consequence is the *Noether Theorem* which states:

If $L_{\mathbf{X}} \mathcal{L} = 0$, then the function

$$\Sigma_0 = \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} , \quad (22)$$

is a constant of motion.

Some comments are necessary at this point. Eq.(22) can be expressed independently of coordinates as a contraction of \mathbf{X} by a Cartan one-form

$$\theta_{\mathcal{L}} \stackrel{\text{def}}{=} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} dq^i . \quad (23)$$

For a generic vector field $\mathbf{Y} = y^i \partial / \partial x^i$, and one-form $\beta = \beta_i dx^i$, we have, by definition, $i_{\mathbf{Y}} \beta = y^i \beta_i$. Thus Eq.(22) can be written as

$$i_{\mathbf{X}} \theta_{\mathcal{L}} = \Sigma_0 . \quad (24)$$

By a point-transformation, the vector field \mathbf{X} becomes

$$\tilde{\mathbf{X}} = (i_{\mathbf{X}} dQ^k) \frac{\partial}{\partial Q^k} + \left(\frac{d}{d\lambda} (i_{\mathbf{X}} dQ^k) \right) \frac{\partial}{\partial \dot{Q}^k} . \quad (25)$$

We see that $\tilde{\mathbf{X}}'$ is still the lift of a vector field defined on the “space of positions.” If \mathbf{X} is a symmetry and we choose a point transformation such that

$$i_{\mathbf{X}}dQ^1 = 1; \quad i_{\mathbf{X}}dQ^i = 0 \quad i \neq 1, \quad (26)$$

we get

$$\tilde{\mathbf{X}} = \frac{\partial}{\partial Q^1}; \quad \frac{\partial \mathcal{L}}{\partial Q^1} = 0. \quad (27)$$

Thus Q^1 is a cyclic coordinate and the dynamics results *reduced* [49, 50].

Furthermore, the change of coordinates given by (26) is not unique and then a clever choice could be very important. In general, the solution of Eq.(26) is not defined on the whole space. It is local in the sense explained above. Besides, it is possible that more than one \mathbf{X} is found, e.g. $\mathbf{X}_1, \mathbf{X}_2$. If they commute, i.e. $[\mathbf{X}_1, \mathbf{X}_2] = 0$, then it is possible to obtain two cyclic coordinates by solving the system

$$i_{\mathbf{X}_1}dQ^1 = 1; i_{\mathbf{X}_2}dQ^2 = 1; i_{\mathbf{X}_1}dQ^i = 0; i \neq 1; i_{\mathbf{X}_2}dQ^i = 0; i \neq 2. \quad (28)$$

The transformed fields will be $\partial/\partial Q^1, \partial/\partial Q^2$. If they do not commute, this procedure is clearly not applicable, since commutation relations are preserved by diffeomorphisms. If the relation $\mathbf{X}_3 = [\mathbf{X}_1, \mathbf{X}_2]$ holds, also \mathbf{X}_3 is a symmetry, being $L_{\mathbf{X}_3}\mathcal{L} = L_{\mathbf{X}_1}L_{\mathbf{X}_2}\mathcal{L} - L_{\mathbf{X}_2}L_{\mathbf{X}_1}\mathcal{L} = 0$. If \mathbf{X}_3 is independent of $\mathbf{X}_1, \mathbf{X}_2$, we can go on until the vector fields close the Lie algebra. The usual approach to this situation is to make a Legendre transformation, going to the Hamiltonian formalism, and then derive the Lie algebra of Poisson brackets.

If we seek for a reduction of dynamics by cyclic coordinates, the procedure is possible in the following way: *i*) we arbitrarily choose one of the symmetries, or a linear combination of them, searching for new coordinates where, as sketched above, the cyclic variables appear. After the reduction, we get a new Lagrangian $\tilde{\mathcal{L}}(\mathbf{Q})$; *ii*) we search again for symmetries in this new configuration space, make a new reduction and so on until possible; *iii*) if the search fails, we try again by another of the existing symmetries.

Let us now assume that \mathcal{L} is of the form (13). As \mathbf{X} is of the form (16), $L_{\mathbf{X}}\mathcal{L}$ will be a homogeneous polynomial of second degree in the velocities plus a inhomogeneous term in the q^i . Since such a polynomial has to be identically zero, each coefficient must be independently zero. If n is the dimension of the configuration space, we get $\{1 + n(n+1)/2\}$ partial differential equations. The system is overdetermined, therefore, if any solution exists, it will be expressed in terms of integration constants instead of boundary conditions. It is also obvious that an overall constant factor in the Lie vector \mathbf{X} is irrelevant. In other words, the Noether Symmetry Approach can be used to select functions which assign the models and such functions (and then the models) can be physically relevant.

Considering the specific case which we are going to discuss, the $f(R)$ cosmology, the situation is the following. The configuration space is $\mathcal{Q} = \{a, R\}$ while the tangent space for the related tangent bundle is $\mathcal{TQ} = \{a, \dot{a}, R, \dot{R}\}$. The Lagrangian is an application

$$\mathcal{L} : \mathcal{TQ} \longrightarrow \mathbb{R} \quad (29)$$

where \mathbb{R} is the set of real numbers. The generator of symmetry is

$$\mathbf{X} = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{R}}. \quad (30)$$

As discussed above, a symmetry exists if the equation $L_{\mathbf{X}}\mathcal{L} = 0$ has solutions. Then there will be a constant of motion on shell, i.e. for the solutions of the Euler equations, as stated above equation (22). In other words, a symmetry exists if at least one of the functions α or β in Eq.(30) is different from zero. As a byproduct, the form of $f(R)$, not specified in the point-like Lagrangian (7), is determined in correspondence to such a symmetry.

IV. NOETHER SYMMETRIES IN $f(R)$ COSMOLOGY

For the existence of a symmetry, we can write the following system of equations (linear in α and β),

$$f_R(\alpha + 2a\partial_a\alpha) + a f_{RR}(\beta + a\partial_a\beta) = 0 \quad (31)$$

$$a^2 f_{RR} \partial_R \alpha = 0 \quad (32)$$

$$2 f_R \partial_R \alpha + f_{RR}(2\alpha + a\partial_a\alpha + a\partial_R\beta) + a\beta f_{RRR} = 0, \quad (33)$$

obtained setting to zero the coefficients of the terms \dot{a}^2 , \dot{R}^2 and $\dot{a}\dot{R}$ in $L_{\mathbf{X}}\mathcal{L} = 0$. In order to make $L_{\mathbf{X}}\mathcal{L} = 0$ vanish we will also look for those particular f 's which, given the Euler dynamics, also satisfy the constraint

$$3\alpha(f - R f_R) - a\beta R f_{RR} - \frac{6\kappa}{a^2}(\alpha f_R + a\beta f_{RR}) + \frac{\rho_{r0}\alpha}{a^4} = 0. \quad (34)$$

This procedure is different from the usual Noether symmetry approach, in the sense that now $L_{\mathbf{X}}\mathcal{L} = 0$ will be solved not for all dynamics (which solve the Euler-Lagrange equations), but only for those f which allows Euler solutions to solve also the constraint (34). Imposing such a constraint on the form of f will turn out to be, as we will show, a sufficient condition to find solutions of the Euler-Lagrange equation which also possess a constant of motion, i.e. a Noether symmetry. As we shall see later on, the system (31), (32) and (33) can be solved exactly. Having a non-trivial solution for α and β for this system, one finds a constant of motion if also the constraint (34) is satisfied. In fact, with these α and β , only those Euler-Lagrange solutions which also satisfy equation (34) will have a constant of motion. However, this will not happen for all $f(R)$'s. The task will be to find such forms of f .

A solution of (31), (32) and (33) exists if explicit forms of α , β are found. If, at least one of them is different from zero, a Noether symmetry exists.

If $f_{RR} \neq 0$, Eq.(32) can be immediately solved being

$$\alpha = \alpha(a). \quad (35)$$

The case $f_{RR} = 0$ is trivial since corresponds to the standard GR. We can rewrite Eqs.(31) and (33) as follows

$$f_R \left(\alpha + 2a \frac{d\alpha}{da} \right) + a f_{RR} (\beta + a \partial_a \beta) = 0 \quad (36)$$

$$f_{RR} \left(2\alpha + a \frac{d\alpha}{da} + a \partial_R \beta \right) + a\beta f_{RRR} = 0. \quad (37)$$

Since $f = f(R)$, then $\partial f / \partial a = 0$, even in the case we consider $R = R(a)$, it is possible to solve equation (37), by writing it as

$$\partial_R(\beta f_{RR}) = -f_{RR} \left(2\frac{\alpha}{a} + \frac{d\alpha}{da} \right) \quad (38)$$

whose general solution can be written as

$$\beta = - \left[\frac{2\alpha}{a} + \frac{d\alpha}{da} \right] \frac{f_R}{f_{RR}} + \frac{h(a)}{f_{RR}}. \quad (39)$$

Therefore one finds that Eq. (36) gives

$$f_R \left[\alpha - a^2 \frac{d^2\alpha}{da^2} - a \frac{d\alpha}{da} \right] + a \left[h - a \frac{dh}{da} \right] = 0, \quad (40)$$

which has solution

$$\alpha = c_1 a + \frac{c_2}{a} \quad \text{and} \quad h = \frac{\bar{c}}{a}, \quad (41)$$

where, being a dimensionless, c_1 and c_2 have the same dimensions. We can further fix α to be dimensionless, this fixes the dimensions of β to be $[\beta] = M^2$. Then also $[\bar{c}] = M^2$, so that we have

$$\beta = - \left[3c_1 + \frac{c_2}{a^2} \right] \frac{f_R}{f_{RR}} + \frac{\bar{c}}{a f_{RR}}. \quad (42)$$

We can now use the expressions for α and β into Eq.(34) as follows

$$f_R = \frac{3a(c_1 a^2 + c_2)f - \bar{c}(a^2 R + 6\kappa)}{2a(c_2 R - 6c_1 \kappa)} + \frac{(c_1 a^2 + c_2)\rho_{r0}}{2a^4(c_2 R - 6c_1 \kappa)}, \quad (43)$$

if $c_2 R - 6\kappa c_1 \neq 0$. It is clear that, for a general f , it will not be possible to solve at the same time the Euler-Lagrange equation and this constraint. Therefore we have to use the Noether constraint in order to find the subset of those

f which make this possible. As we shall see later, it is convenient to look for a parametric solution in the form $[H(a), f(R(a))]$. In this case, since $f_R = f'/R'$, the Noether condition corresponds to the following ODE

$$\frac{f'(a)}{R'(a)} = \frac{3a(c_1 a^2 + c_2)f(a) - \bar{c}(a^2 R(a) + 6\kappa)}{2a(c_2 R(a) - 6c_1 \kappa)} + \frac{(c_1 a^2 + c_2)\rho_{r0}}{2a^4(c_2 R(a) - 6c_1 \kappa)}. \quad (44)$$

It should be noted that this change of variable is defined only if $R' \neq 0$, that is if R is not constant during the evolution. When this happens Eq. (34) or (45) sets $a = a_0 = \text{constant}$, which corresponds to an uninteresting solution.

Any Euler-Lagrange solution, by definition, satisfies the Einstein equations. However we will show that there are forms of $f(R)$, for which a subset of those solution will also be a Noether solution. In fact, Eq.(43) can also be rewritten as

$$c_1 a^2 (\rho_{r0} + 3a^4 f + 12\kappa a^2 f_R) + c_2 [\rho_{r0} + a^4 (3f - 2R f_R)] = \bar{c} a^3 (a^2 R + 6\kappa). \quad (45)$$

Therefore we look for a family of solutions that, being a Noether symmetry, gives a class of $f(R)$ models.

This symmetry implies the existence of the following constant of motion

$$\alpha (6 f_{RR} a^2 \dot{R} + 12 f_R a \dot{a}) + \beta (6 f_{RR} a^2 \dot{a}) = 6 \mu_0^3 = \text{constant}, \quad (46)$$

where μ_0 has the dimensions of a mass. Equation (46) can be recast in the form

$$\frac{d(f_R)}{dt} = f_{RR} \dot{R} = \frac{\mu_0^3}{a(c_1 a^2 + c_2)} + \frac{c_1 a^2 - c_2}{c_1 a^2 + c_2} f_R H - \frac{\bar{c} a}{c_1 a^2 + c_2} H, \quad (47)$$

or, using the time-parameter a

$$aH(a) \left(\frac{f''(a)}{R'(a)} - \frac{f'(a)R''(a)}{R'(a)^2} \right) - \frac{(a^2 c_1 - c_2)H(a)f'(a)}{(c_1 a^2 + c_2)R'(a)} = \frac{\mu_0^3}{a(c_1 a^2 + c_2)} - \frac{\bar{c} a}{c_1 a^2 + c_2} H(a). \quad (48)$$

Once Eq. (44) is solved, because the Noether constraint is satisfied, the solution $[H(a), f(R(a))]$ will automatically solve also (48) for a particular μ_0 . Equation (46) can be used to reduce the order of the Friedmann equation. In fact, writing Eq.(8) as

$$f - 6 f_{RR} \dot{R} H - 6 f_R H^2 - f_R \left(R + \frac{6\kappa}{a^2} \right) - \frac{\rho_{m0}}{a^3} - \frac{\rho_{r0}}{a^4} = 0, \quad (49)$$

we have

$$f - \frac{12 c_1 a^2}{c_1 a^2 + c_2} f_R H^2 - f_R \left(R + \frac{6\kappa}{a^2} \right) + \frac{6 \bar{c} a}{c_1 a^2 + c_2} H^2 = \frac{6 \mu_0^3 H}{a(c_1 a^2 + c_2)} + \frac{\rho_{m0}}{a^3} + \frac{\rho_{r0}}{a^4}, \quad (50)$$

where f_R is given by (43). We will use this relation in order to find out exact cosmological solutions. Namely, we will search for solutions depending on the constant of motion μ_0 determined by the Noether symmetry.

V. EXACT COSMOLOGICAL SOLUTIONS

In order to find out exact cosmological solutions, let us discuss the Noether condition Eq.(45) and the dynamical system (8),(10) with respect to the values of the integration constants $c_{1,2}$, the structural parameters k, ρ_{r0}, ρ_{m0} and the Noether charge μ_0 . Beside cosmological solutions, also the explicit form of $f(R)$ will result fixed in the various cases. As we shall see later on, analytical solutions can be easily found for the case when both \bar{c} and μ_0 vanish at the same time. Therefore in all this section, except one subsection, we will set $\bar{c} = 0$.

A. Case $c_1 = 0$

In this case, the Noether condition (45) reduces to

$$2 f_R - 3 f = \frac{\rho_{r0}}{a^4}. \quad (51)$$

1. Vacuum and pure dust case

In vacuum, or in the presence of dust only, i.e. $\rho_{r0} = 0$, we find

$$f = f_0 \left(\frac{R}{R_0} \right)^{3/2}. \quad (52)$$

This solution, for the vacuum case $\rho_{r0} = \rho_{m0} = 0$, has been already found [48]. The absence of a ghost imposes that $f_R < 0$, i.e. $f_0 > 0$ since $R_0 < 0$. In the case of dust and no radiation ($\rho_{m0} \neq 0, \rho_{r0} = 0$), one can substitute Eq.(52) into (50) to find

$$\left(\frac{R}{R_0} \right)^{3/2} + \frac{18\kappa}{a^2 R_0} \left(\frac{R}{R_0} \right)^{1/2} = -\frac{12\mu_0^3 H}{c_2 a f_0} - \frac{2\rho_{m0}}{a^3 f_0}. \quad (53)$$

1. $k = 0$. In this case, for consistency, we need the right hand side of (53) to be positive. If $\mu_0 = 0$ (case for which analytical solutions could be given), this is impossible as $f_0 > 0$, therefore there is no ghost-free solution. For the more general case $\mu_0^3/c_2 < 0$, there could be a physical solution: the non-linearity of the equations does not allow us to find analytical solutions for this case. Nevertheless, solutions (to be found numerically) may still exist.
2. $k \neq 0$. The Ricci scalar can be found as the solution of Eq. (53). For $\mu_0 = 0$, we have a cubic equation in $(R/R_0)^{1/2}$, for which a real solution always exists (which may not be positive though). Looking at equation (53), the case $\mu_0 = 0, k = -1$ has no ghost-free solutions ($f_0 < 0$). Also the case $\mu_0 = 0, k = 1$ has no solution, because we have

$$\sqrt{\frac{R}{R_0}} = \left[\frac{\tilde{B}_0^{1/3}}{f_0 R_0} - \frac{6\kappa f_0}{\tilde{B}_0^{1/3}} \right] \frac{1}{a}, \quad (54)$$

where we have defined the constant

$$\tilde{B}_0 = \sqrt{f_0^4 \rho_{m0}^2 R_0^6 + 216 f_0^6 \kappa^3 R_0^3 - f_0^2 \rho_{m0} R_0^3}, \quad (55)$$

which implies that $(f_0/\rho_{m0})^2 (\kappa/R_0)^3 > -1/216$. If so, then, since $R_0 < 0, \tilde{B}_0 > 0$. However, this would lead to a negative value for $(R/R_0)^{1/2}$.

2. Dust and radiation case

In this case we have

$$f_R = \frac{3}{2} \frac{f}{R} + \frac{\rho_{r0}}{2a^4 R}. \quad (56)$$

Once again, in order to have $f_R < 0$, and $R < 0$ during the evolution of the universe one requires

$$f > -\frac{\rho_{r0}}{3a^4}. \quad (57)$$

If we substitute the expression for f_R into the reduced Friedmann Eq.(50) we find

$$f = -\frac{12\mu_0^3 a H R}{c_2 (R a^2 + 18\kappa)} - \frac{6\kappa \rho_{r0}}{a^4 (R a^2 + 18\kappa)} - \frac{3\rho_{r0} R}{a^2 (R a^2 + 18\kappa)} - \frac{2\rho_{m0} R}{a (R a^2 + 18\kappa)}. \quad (58)$$

This relation gives f as a function of a being $R = R(a)$. It has to be $c_2 \neq 0$ otherwise the Noether condition becomes trivial. This expression can be inserted back into (56). Assuming $R = R(a)$ as a monotonic function of a , one finds that $f_R = (df/da)/(dR/da)$, and equation (51) becomes a differential equation for $R(a)$, which can be written as

$$R' = \frac{6}{a^3 (18a^3 H \mu_0^3 + 4c_2 \rho_{r0} + 3ac_2 \rho_{m0}) (R a^2 + 6\kappa)} \times \{ -R^2 [2a^3 (H - aH') \mu_0^3 + c_2 (2\rho_{r0} + a\rho_{m0})] a^4 + 6\kappa R [6a^3 \mu_0^3 (H + aH') - c_2 (4\rho_{r0} + a\rho_{m0})] a^2 - 72c_2 \kappa^2 \rho_{r0} \}, \quad (59)$$

where the prime denotes differentiation with respect to the scale factor a . Eq.(59) can be further rewritten as a second order differential equation in $H(a)$, by using equation (10),

$$R = -12 H^2 - 6 a H H' - 6 \frac{\kappa}{a^2}. \quad (60)$$

Substituting (60) into (59) one finds

$$\begin{aligned} H'' = & -\frac{1}{a^4 H^2 (18a^3 H \mu_0^3 + 4c_2 \rho_{r0} + 3ac_2 \rho_{m0})} \times \{24a\kappa^2 \mu_0^3 + H[a^2\{6a^3 H \mu_0^3 + 4c_2 \rho_{r0} \\ & + 3ac_2 \rho_{m0}\}H'^2 + a[12a\kappa \mu_0^3 + H(78a^3 H \mu_0^3 + 32c_2 \rho_{r0} + 21ac_2 \rho_{m0})]H' \\ & + 12H[2a\kappa \mu_0^3 + H(2a^3 H \mu_0^3 + 2c_2 \rho_{r0} + ac_2 \rho_{m0})]]\} - 8c_2 \kappa \rho_{r0} \}. \end{aligned} \quad (61)$$

This differential equation selects those $f(R)$ models which satisfy, at the same time, both the Friedmann equation and the Noether condition. It has to be stressed that, having chosen a as the time variable, finding the $H(a)$'s which solve (61) uniquely fixes the metric tensor. Hence, $H(a)$ represents a fully solved exact solution for the Einstein equations. Of course, if one wants to know the link between a and the proper time, $a = a(t)$, one needs to find the integral $t = \int da/(aH)$.

The case $\mu_0 = 0$ is interesting as it allow us to find analytical solutions, as the differential equation becomes (2nd order and) linear for the variable H^2 . In this case, the solution of the equation will be a family $H = H(a, d_1, d_2, c_2, \mu_0, \kappa, \rho_{r0}, \rho_{m0})$, where $d_{1,2}$ are two constants coming from the integration of Eq.(61). In turn, by using Eq.(60), it is possible to define a function $R = R(a, d_1, d_2, c_2, \mu_0, \rho_{r0}, \rho_{m0})$, which can then be substituted into Eq.(58) in order to find the explicit parametric form of $f(R)$, i.e. $f = f(a, d_1, d_2, c_2, \mu_0, \rho_{r0}, \rho_{m0})$. In other words, we find the explicit parametric form for $f(R)$ where the parameter used to describe the $f(R)$ is the scale factor a (see also [21] for a comparison with observations. However, in that case, the adopted $f(R)$ models were constructed by phenomenological considerations and not derived from some first principle, as the existence of symmetries as discussed here).

We can distinguish some relevant cases.

1. $k = 0, \mu_0 = 0$. In this case, by exactly integrating equation (61), we find

$$H^2 = d_2 \frac{d_1 + 8a \rho_{r0} + 3 \rho_{m0} a^2}{a^4}, \quad (62)$$

where $d_{1,2}$ are integration constants, with $[d_1] = M^4$ and $[d_2] = M^{-2}$. This expression for $H(a)$ together with (58) and (60) form a solution for the set of ODE's (9), and (44), so that Eq. (48) is satisfied giving $\mu_0 = 0$. Although this solution is analytical it cannot be accepted because it allows for a negative Newton constant. In fact, equation (57) cannot be satisfied by equation (58) if $k = 0, \mu_0 = 0$. However the non-linear case $\mu_0/c_2 < 0$ could actually lead to physical solutions (to be discussed elsewhere in a forthcoming paper). For the same reason, also the case $k = -1, \mu_0 = 0$ should be rejected.

2. $k = 1, \mu_0 = 0$. As far as $R < -18\kappa/a^2$, the second term in the l.h.s. of equation (58) becomes positive, allowing for the possibility of finding a physical solution. The integration of (61) leads to

$$H^2 = \left(\sqrt{2} d_1 - \frac{32 \rho_{r0}^2 \kappa}{9 \rho_{0m}^2} \right) \frac{1}{a^4} + \left(8 d_2 \rho_{r0} - \frac{16 \rho_{r0} \kappa}{3 \rho_{m0}} \right) \frac{1}{a^3} + \frac{3 d_2 \rho_{m0}}{a^2}, \quad (63)$$

with $[d_1] = M^2$, and $[d_2] = M^{-2}$. In order to find d_1 and d_2 one can fit this formula with the standard Friedmann equation of GR with only matter, radiation and curvature. Therefore, one has to consider

$$\sqrt{2} d_1 - \frac{32 \rho_{r0}^2 \kappa}{9 \rho_{0m}^2} = H_0^2 \Omega_{r0}^{\text{eff}}, \quad (64)$$

$$8 d_2 \rho_{r0} - \frac{16 \rho_{r0} \kappa}{3 \rho_{m0}} = H_0^2 \Omega_{m0}^{\text{eff}}, \quad (65)$$

$$3 d_2 \rho_{m0} = H_0^2 \Omega_{k0}^{\text{eff}}, \quad (66)$$

but this system admits no solutions as one finds

$$\kappa = \frac{1}{2} H_0^2 \Omega_{k0}^{\text{eff}} - \frac{3}{16} \frac{\rho_{m0}}{\rho_{r0}} H_0^2 \Omega_{m0}^{\text{eff}} < 0 \quad (67)$$

using today's data [52].

B. Case $c_2 = 0$

In this case, the Noether condition (45) reduces to

$$\rho_{r0} + 3a^4 f + 12\kappa a^2 f_R = 0. \quad (68)$$

1. Vacuum and dust only case

In this case we have $\rho_{r0} = 0$, and a flat universe cannot be solution as one would obtain $f = 0$. Considering $k \neq 0$ one finds

$$f_R = -\frac{a^2 f}{4\kappa}. \quad (69)$$

Since $f_R < 0$ then f is positive when $k < 0$ and viceversa. Substituting this into the Friedmann equation one finds

$$\{a^3 c_1 [(12H^2 + R)a^2 + 10\kappa]\} f = 4\kappa (6H\mu_0^3 + c_1 \rho_{m0}). \quad (70)$$

Restricting ourselves only to the study of the simple and linear case of a vanishing μ_0 , we can distinguish two cases

1. $\rho_{m0} = 0, \mu_0 = 0$. In this case one needs to impose

$$R = -12H^2 - 10\frac{\kappa}{a^2}, \quad (71)$$

which, together with the definition of R , gives

$$H^2 = 2d_1 - \frac{2\kappa}{3a^2}, \quad (72)$$

where d_1 is a constant of integration with dimensions M^2 . This behavior describes a universe with only a cosmological constant and curvature. Equation (68) can now be solved for $f(a)$ giving

$$f = \frac{d_2}{a} = d_2 \left[-\frac{R + 24d_1}{2\kappa} \right]^{1/2}, \quad (73)$$

where d_2 is a constant of integration with dimensions M^4 .

2. $\rho_{m0} \neq 0, \mu_0 = 0$. In this case the Friedmann equation and (69) give

$$f = \frac{4\kappa \rho_{m0}}{(12H^2 + R)a^5 + 10\kappa a^3}. \quad (74)$$

Substituting this expression in (69), and using the definition for R in terms of $H(a)$ one finds a linear 2nd order differential equation in $H^2(a)$, which has solution

$$H^2 = \frac{d_1}{2a^4} + 2d_2 - \frac{2\kappa}{3a^2}, \quad (75)$$

where $d_{1,2}$ are integration constants, and $[d_1] = [d_2] = M^2$. Therefore one has

$$R = -24d_2 - \frac{2\kappa}{a^2}, \quad (76)$$

$$f = -\frac{2\kappa \rho_{m0}}{3a d_1}. \quad (77)$$

2. Radiation and dust case

Also in this case, we have three possibilities, according to the values of k .

1. $k = 0$. In this case one finds that

$$f = -\frac{\rho_{r0}}{3a^4}. \quad (78)$$

Therefore we have

$$f_R = \frac{f'}{R'} = \frac{4}{3} \frac{\rho_{r0}}{a^5 R'}. \quad (79)$$

A well-behaved background evolution requires, with our conventions, $R' > 0$, so that $f_R > 0$. This means a negative effective Newton constant, i.e. the solution cannot be accepted.

2. $k \neq 0$. In this case, using equation (68) one finds

$$f_R = -\frac{\rho_{r0}}{12\kappa a^2} - \frac{f a^2}{4\kappa}, \quad (80)$$

and then using Friedmann equation (50) one can solve for f , as follows

$$f = \frac{-c_1 (12H^2 + R) \rho_{r0} a^2 + 12\kappa (6H\mu_0^3 + c_1 \rho_{m0}) a + 6c_1 \kappa \rho_{r0}}{3a^4 c_1 [(12H^2 + R) a^2 + 10\kappa]}. \quad (81)$$

By plugging this relation into the Noether condition (68), and using the definition of R in terms of H, H' , and a , one finds the following differential equation for $H(a)$

$$\begin{aligned} H'' = & \frac{aH \left[- (18aH\mu_0^3 + 3ac_1\rho_{m0} + 4c_1\rho_{r0}) H'^2 a^4 - 3 (aH (30aH\mu_0^3 + 5ac_1\rho_{m0} + 8c_1\rho_{r0}) - 4\kappa\mu_0^3) H' a^2 \right.}{a^5 H^2 (18aH\mu_0^3 + 3ac_1\rho_{m0} + 4c_1\rho_{r0})} \\ & \left. + \frac{4\kappa (6aH\mu_0^3 + ac_1\rho_{m0} + 2c_1\rho_{r0})}{a^5 H^2 (18aH\mu_0^3 + 3ac_1\rho_{m0} + 4c_1\rho_{r0})} \right] - 8\kappa^2 \mu_0^3}{a^5 H^2 (18aH\mu_0^3 + 3ac_1\rho_{m0} + 4c_1\rho_{r0})}. \end{aligned} \quad (82)$$

In the case $\mu_0 = 0, \rho_{m0} \neq 0$, this differential equation can be exactly integrated to give

$$H^2 = \frac{256\kappa\rho_{r0}^3}{405a^5\rho_{m0}^3} + \frac{16\kappa\rho_{r0}^2}{27a^4\rho_{m0}^2} + \frac{8d_1\rho_{r0}}{5a^5} - \frac{2\kappa}{3a^2} + \frac{3\rho_{m0}d_1}{2a^4} + 2d_2, \quad (83)$$

where $d_{1,2}$ are two constants of integration with dimensions $[d_1] = M^{-2} = [d_2]^{-1}$. It is interesting to note the presence of a new cosmological term in this Friedmann equation, which goes as a^{-5} , which would correspond to a matter term with equation of state parameter $w = 2/3$.

If $\mu_0 = 0, \rho_{m0} = 0$, i.e. a universe filled with radiation only, equation (82) has the following solution

$$H^2 = 2d_2 + \frac{2d_1}{5a^5} - \frac{2\kappa}{3a^2}, \quad (84)$$

with $[d_1] = [d_2] = M^2$.

C. Case $c_1, c_2 \neq 0$

In this case, one can divide equation (45) by c_1 finding

$$f_R = \frac{a^2 + c_3}{c_3 R - 6k} \frac{\rho_{r0} + 3a^4 f}{2a^4}, \quad (85)$$

where $c_3 = c_2/c_1 \neq 0$. This implies that

$$f_{RR} \dot{R} = \frac{\tilde{\mu}_0^3}{a(a^2 + c_3)} + \frac{a^2 - c_3}{a^2 + c_3} f_R H, \quad (86)$$

where $\tilde{\mu}_0^3 = \mu_0^3/c_1$.

Friedmann equation Eq.(50) can be rewritten as

$$f - \frac{12a^2}{a^2 + c_3} f_R H^2 - f_R \left(R + \frac{6k}{a^2} \right) = \frac{6\tilde{\mu}_0^3 H}{a(a^2 + c_3)} + \frac{\rho_{m0}}{a^3} + \frac{\rho_{r0}}{a^4}. \quad (87)$$

By substituting (85) into (87), and solving for f , one finds

$$f = \frac{12\tilde{\mu}_0^3 a^5 H (6k - c_3 R)}{a^4 (a^2 + c_3) [3(12H^2 + R)a^4 + (30k + c_3 R)a^2 + 18c_3 k]} - \frac{\rho_{r0} (12H^2 + R)a^4 + 2\rho_{m0} (c_3 R - 6k)a^3 + 3\rho_{r0} (c_3 R - 2k)a^2 + 6c_3 k \rho_{r0}}{a^4 [3(12H^2 + R)a^4 + (30k + c_3 R)a^2 + 18c_3 k]}, \quad (88)$$

which means that the Noether symmetry, combined with the dynamics, determines the form of f . In this case f is a function of a since both R and H are functions of a . We can still go further by using the same trick used in the previous section, i.e. considering f as an implicit function of a into the Noether condition (85). Since $f = f(R(a))$ one finds

$$f_R = \frac{df}{dR} = \frac{da}{dR} \frac{df}{da} = \frac{f'}{R'}. \quad (89)$$

Plugging Eqs.(88) and (89) into (85), one finds a second order differential equation for H , as follows

$$H'' = \frac{1}{a^4(a^2 + c_3)(3a^2 + c_3)H^2 [18\tilde{\mu}_0^3 H a^3 + (a^2 + c_3)(4\rho_{r0} + 3a\rho_{m0})]} \times \{ -24c_3(3a^2 + c_3)\tilde{\mu}_0^3 H^4 a^5 - 24(a^2 + c_3)^2 k^2 \tilde{\mu}_0^3 a - H^2 [6(3a^2 + c_3)^2 \tilde{\mu}_0^3 H'^2 a^4 + 24(-3a^4 - 2c_3 a^2 + c_3^2)k \tilde{\mu}_0^3 + (a^2 + c_3)^2 (45\rho_{m0} a^3 + 72\rho_{r0} a^2 + 21c_3 \rho_{m0} a + 32c_3 \rho_{r0}) H'] a^3 - 6H^3 [(3a^2 + c_3)(15a^2 + 13c_3)\tilde{\mu}_0^3 H' a^4 + 2c_3(a^2 + c_3)^2 (2\rho_{r0} + a\rho_{m0})] a^2 - (a^2 + c_3)H [a^4 H' [12(c_3 - 3a^2)k \tilde{\mu}_0^3 + (a^2 + c_3)(3a^2 + c_3)(4\rho_{r0} + 3a\rho_{m0}) H'] - 4(a^2 + c_3)k(3\rho_{m0} a^3 + 6\rho_{r0} a^2 + 2c_3 \rho_{r0})] \}. \quad (90)$$

This differential equation defines the dynamics of the Noether solutions for a generic $f(R)$ model compatible with the Noether symmetry. This result is relevant since there is a free parameter c_3 , which together with the initial conditions for H_0 and H'_0 , uniquely specify the dynamics. This non-linear ODE is still of second order in $H(a)$ as the 0, 0-Einstein equation for any $f(R)$ theory. However, there is a huge improvement as this equation is independent of the explicit form $f(R)$, having as the only unknown parameters two real numbers, c_3 and μ_0 , the Noether charge. This also says that for any value of the Noether charge there is a solution, the solution of (90). Therefore all the solutions of (90), as c_3, μ_0 vary, represent the whole set of Noether-charged cosmological solutions of the $f(R)$ theories.

1. Vacuum and pure dust case

In this case equation (85) reduces to

$$f_R = \frac{3f(a^2 + c_3)}{2(Rc_3 - 6\kappa)}, \quad (91)$$

whereas f can be written as

$$f = \frac{2(6\kappa - Rc_3)((6H\mu_0^3 + \rho_{m0})a^2 + \rho_{m0}c_3)}{a(a^2 + c_3)(3(12H^2 + R)a^4 + (30\kappa + Rc_3)a^2 + 18\kappa c_3)}. \quad (92)$$

The case $\rho_{m0} = 0, \mu_0 = 0$ admits no solutions, therefore, as before, we will only discuss the case $\mu_0 = 0, \rho_{m0} \neq 0$, for which we can recast f in the following form

$$f = 3(12H^2 + R)a^4 + (30\kappa + Rc_3)a^2 + 18\kappa c_3. \quad (93)$$

Inserting this relation into (91) together with the definition of R one finds

$$H'' = \frac{-4c_3 H^2 - a(15a^2 + 7c_3) H' H - a^2(3a^2 + c_3) H'^2 + 4\kappa}{a^2(3a^2 + c_3) H}, \quad (94)$$

whose general solution reads

$$H^2 = -\frac{c_3 \kappa}{9a^4} - \frac{2\kappa}{3a^2} + \frac{2d_1}{a^4} + \frac{2c_3 d_2}{a^2} + 3d_2. \quad (95)$$

2. Pure radiation case

Once again, studying Eq. (90) to the case $\mu_0 = 0$ and $\rho_{m0} = 0$, we find the following equation

$$(H^2)'' = -\frac{18a^2 + 8c_3}{a(3a^2 + c_3)} (H^2)' - \frac{12c_3 H^2}{a^2(3a^2 + c_3)} + \frac{2k(6a^2 + 2c_3)}{a^4(3a^2 + c_3)}. \quad (96)$$

The general solution, when $c_3 > 0$, for this ODE is

$$\begin{aligned} H^2 = & \frac{3c_3 d_1}{a^4} + \frac{27d_1}{c_3} + \frac{18d_1}{a^2} + \frac{5\sqrt{3}\sqrt{c_3}d_2}{a^3} + \frac{9\sqrt{3}d_2}{a\sqrt{c_3}} + \frac{4\kappa}{c_3} + \frac{2\kappa}{a^2} \\ & + \frac{3c_3 d_2 \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right)}{a^4} + \frac{27d_2 \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right)}{c_3} + \frac{18d_2 \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right)}{a^2}, \end{aligned} \quad (97)$$

whereas, for $c_3 < 0$, one finds

$$\begin{aligned} H^2 = & \frac{3c_3 d_1}{a^4} + \frac{27d_1}{c_3} + \frac{18d_1}{a^2} - \frac{5\sqrt{3}\sqrt{c_3}d_2}{a^3} + \frac{9\sqrt{3}d_2}{a\sqrt{-c_3}} + \frac{4\kappa}{c_3} + \frac{2\kappa}{a^2} \\ & + \frac{3c_3 d_2 \operatorname{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right)}{a^4} + \frac{27d_2 \operatorname{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right)}{c_3} + \frac{18d_2 \operatorname{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right)}{a^2}. \end{aligned} \quad (98)$$

Either expression for $H(a)$ together with Eq. (88) and Eq. (60) form a solution for (9), and (44), and possess $\mu_0 = 0$ Noether charge.

3. Matter and Radiation case

Let us restrict our study to the case $\tilde{\mu} = 0$, for which we can find analytical solutions. Eq.(90) reduces to

$$\begin{aligned} (H^2)'' = & -\frac{(45\rho_{m0}a^3 + 72\rho_{r0}a^2 + 21c_3\rho_{m0}a + 32c_3\rho_{r0})}{a(3a^2 + c_3)(4\rho_{r0} + 3a\rho_{m0})} (H^2)' \\ & - \frac{24c_3(\rho_{m0}a + 2\rho_{r0})H^2}{a^2(3a^2 + c_3)(4\rho_{r0} + 3a\rho_{m0})} + \frac{8k(3\rho_{m0}a^3 + 6\rho_{r0}a^2 + 2c_3\rho_{r0})}{a^4(3a^2 + c_3)(4\rho_{r0} + 3a\rho_{m0})}. \end{aligned} \quad (99)$$

It is remarkable that this differential equation is linear in H^2 . This makes the problem of solving it much easier. In fact, analytical solutions for $k = 0, \pm 1$ can be achieved. Let us discuss them.

1. $k = 0$. The solution of Eq.(99) is

$$\begin{aligned} H^2 = & \frac{4d_1 d_2 c_3^{9/2}}{a^4} + \frac{24d_1 d_2 c_3^{7/2}}{a^2} - \frac{\rho_{0m} d_2 c_3^{5/2}}{a^4} + 36d_1 d_2 c_3^{5/2} \\ & + \frac{2\sqrt{3}\rho_{r0} \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right) d_2 c_3^2}{a^4} + \frac{10\rho_{r0} d_2 c_3^{3/2}}{a^3} + \frac{12\sqrt{3}\rho_{r0} \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right) d_2 c_3}{a^2} \\ & + \frac{18\rho_{r0} d_2 \sqrt{c_3}}{a} + 18\sqrt{3}\rho_{r0} \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right) d_2, \end{aligned} \quad (100)$$

where d_1 and d_2 are integration constants with dimensions, $[d_1] = M^4$, and $[d_2] = M^{-2}$. This is clearly a deviation from standard GR, because there is a $1/a$ term, which leads to an accelerated behavior if dominates. Furthermore there are terms, all involving ρ_{r0} , which include the arctangent of a , where c_3 is supposed to be positive. These terms have different behavior at low and high redshift. In fact since $\lim_{a \rightarrow 0} \arctan(a) \sim a$ at high redshifts, these terms behave as dust, $1/a$ and a respectively, and are subdominant with respect to the radiation. On the other hand, since $\lim_{a \rightarrow \infty} \arctan(a) \sim \pi/2$ for large and positive a , these terms will behave as radiation, curvature and cosmological constant respectively. It is also interesting to notice that in order to have a true dust matter component at late times, it has to be

$$10 \rho_{r0} d_2 c_3^{3/2} = \frac{8\pi G}{3} \rho_{m0}. \quad (101)$$

This means that ρ_{r0} behaves as the source of matter component in this modified Friedmann equation. A cosmological constant term is also present. It is determined by the integration constants of the Noether condition.

As for the case $c_3 < 0$, the solution of Eq. (99) can be written as follows

$$\begin{aligned} H^2 = & -\frac{4d_1 d_2 (-c_3)^{9/2}}{a^4} + \frac{24d_1 d_2 (-c_3)^{7/2}}{a^2} + \frac{\rho_{m0} d_2 (-c_3)^{5/2}}{a^4} - 36d_1 d_2 (-c_3)^{5/2} \\ & + \frac{2\sqrt{3}\rho_{r0} \operatorname{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right) d_2 c_3^2}{a^4} + \frac{10\rho_{r0} d_2 (-c_3)^{3/2}}{a^3} + \frac{12\sqrt{3}\rho_{r0} \operatorname{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right) d_2 c_3}{a^2} \\ & - \frac{18\rho_{r0} d_2 \sqrt{-c_3}}{a} + 18\sqrt{3}\rho_{r0} \operatorname{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right) d_2. \end{aligned} \quad (102)$$

For this solution, as a pedagogical example, more detailed calculations and a link with scalar-tensor theories are given in the appendix.

2. $k \neq 0$. The general solution is

$$\begin{aligned} H^2 = & -\frac{32\kappa \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right) \rho_{r0}^3}{9\sqrt{3}a^4 \rho_{m0}^3 \sqrt{c_3}} - \frac{160\kappa \rho_{r0}^3}{27a^3 \rho_{m0}^3 c_3} - \frac{64\kappa \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right) \rho_{r0}^3}{3\sqrt{3}a^2 \rho_{m0}^3 c_3^{3/2}} - \frac{32\kappa \rho_{r0}^3}{3a \rho_{m0}^3 c_3^2} - \frac{32\kappa \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right) \rho_{r0}^3}{\sqrt{3} \rho_{m0}^3 c_3^{5/2}} \\ & - \frac{16\kappa \rho_{r0}^2}{3a^2 \rho_{m0}^2 c_3} - \frac{8\kappa \rho_{r0}^2}{27a^4 \rho_{m0}^2} - \frac{8\kappa \rho_{r0}^2}{\rho_{m0}^2 c_3^2} + \frac{\sqrt{3} \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right) d_2 \rho_{r0}}{2a^4 c_3^{5/2}} + \frac{5d_2 \rho_{r0}}{2a^3 c_3^3} + \frac{3\sqrt{3} \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right) d_2 \rho_{r0}}{a^2 c_3^{7/2}} \\ & + \frac{9d_2 \rho_{r0}}{2ac_3^4} + \frac{9\sqrt{3} \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right) d_2 \rho_{r0}}{2c_3^{9/2}} - \frac{2\kappa \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right) \sqrt{c_3} \rho_{r0}}{\sqrt{3}a^4 \rho_{m0}} - \frac{4\sqrt{3}\kappa \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right) \rho_{r0}}{a^2 \rho_{m0} \sqrt{c_3}} \\ & - \frac{10\kappa \rho_{r0}}{3a^3 \rho_{m0}} - \frac{6\kappa \rho_{r0}}{a \rho_{m0} c_3} - \frac{6\sqrt{3}\kappa \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right) \rho_{r0}}{\rho_{m0} c_3^{3/2}} - \frac{2\kappa}{3a^2} - \frac{\kappa c_3}{9a^4} + \frac{6d_1}{a^2 c_3} + \frac{9d_1}{c_3^2} + \frac{d_1}{a^4} - \frac{\rho_{m0} d_2}{4a^4 c_3^2}. \end{aligned} \quad (103)$$

Also in these cases we have interesting behaviors matching the main cosmological eras. The integration constants $d_{1,2}$ have dimensions respectively $[d_1] = M^2$, and $[d_2] = M^{-2}$. The analysis, for both this and the previous case ($k = 0$), of the set of parameters $\{d_1, d_2, c_3\}$ which can be bounded by observations will be done in a forthcoming paper.

Eq. (99), for the case $c_3 < 0$, has solution

$$\begin{aligned} H^2 = & \frac{32\kappa \operatorname{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right) \rho_{r0}^3}{9\sqrt{3}a^4 \rho_{m0}^3 \sqrt{-c_3}} - \frac{160\kappa \rho_{r0}^3}{27a^3 \rho_{m0}^3 c_3} - \frac{64\kappa \operatorname{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right) \rho_{r0}^3}{3\sqrt{3}a^2 \rho_{m0}^3 (-c_3)^{3/2}} - \frac{32\kappa \rho_{r0}^3}{3a \rho_{m0}^3 c_3^2} + \frac{32\kappa \operatorname{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right) \rho_{r0}^3}{\sqrt{3} \rho_{m0}^3 (-c_3)^{5/2}} \\ & - \frac{16\kappa \rho_{r0}^2}{3a^2 \rho_{m0}^2 c_3} - \frac{8\kappa \rho_{r0}^2}{27a^4 \rho_{m0}^2} - \frac{8\kappa \rho_{r0}^2}{\rho_{m0}^2 c_3^2} - \frac{\sqrt{3} \operatorname{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right) d_2 \rho_{r0}}{2a^4 (-c_3)^{5/2}} + \frac{5d_2 \rho_{r0}}{2a^3 c_3^3} + \frac{3\sqrt{3} \operatorname{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right) d_2 \rho_{r0}}{a^2 (-c_3)^{7/2}} \\ & + \frac{9d_2 \rho_{r0}}{2ac_3^4} - \frac{9\sqrt{3} \operatorname{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right) d_2 \rho_{r0}}{2(-c_3)^{9/2}} - \frac{2\kappa \operatorname{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right) \sqrt{-c_3} \rho_{r0}}{\sqrt{3}a^4 \rho_{m0}} + \frac{4\sqrt{3}\kappa \operatorname{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right) \rho_{r0}}{a^2 \rho_{m0} \sqrt{-c_3}} \\ & - \frac{10\kappa \rho_{r0}}{3a^3 \rho_{m0}} - \frac{6\kappa \rho_{r0}}{a \rho_{m0} c_3} - \frac{6\sqrt{3}\kappa \operatorname{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right) \rho_{r0}}{\rho_{m0} (-c_3)^{3/2}} - \frac{2\kappa}{3a^2} - \frac{\kappa c_3}{9a^4} + \frac{6d_1}{a^2 c_3} + \frac{9d_1}{c_3^2} + \frac{d_1}{a^4} - \frac{\rho_{m0} d_2}{4a^4 c_3^2}. \end{aligned} \quad (104)$$

It is worthy to note that once the free parameters are constrained by the data (the set of allowed parameters might be empty anyhow), one can select physically interesting $f(R)$ models as in [21].

4. Non-linear case, $\tilde{\mu}_0 \neq 0$

In this more general case, Eq.(90) cannot be written as a linear differential equation in H^2 , therefore it is not possible to achieve an analytical general solution. However, after fixing initial conditions for H and giving suitable values for the parameters, one can solve it numerically. These initial conditions fix, in turn, the $f(R)$ model and the behavior of $H(a)$.

5. General non-linear case, $\bar{c} \neq 0$ and $\tilde{\mu}_0 \neq 0$

By using Eq. (43) inside Eq. (50) one finds the following expression for f

$$f = \frac{c_1 \bar{c} R (12H^2 + R) a^5}{(c_1 a^2 + c_2) \Delta} + \frac{\bar{c} R (12c_2 H^2 + 12c_1 \kappa + c_2 R) a^3}{(c_1 a^2 + c_2) \Delta} + \frac{2(36c_1 \kappa H \mu_0^3 - 6c_2 H R \mu_0^3 + 18c_1 \bar{c} \kappa^2 + 6c_1^2 \kappa \rho_{m0} + 6c_2 \bar{c} \kappa R - c_1 c_2 \rho_{m0} R) a}{(c_1 a^2 + c_2) \Delta} - \frac{2c_2 (-18\bar{c} \kappa^2 - 6c_1 \rho_{m0} \kappa + c_2 \rho_{m0} R)}{(c_1 a^2 + c_2) \Delta a} - \frac{\rho_{r0} (12c_1 H^2 a^4 + c_1 R a^4 - 6c_1 \kappa a^2 + 3c_2 R a^2 + 6c_2 \kappa)}{\Delta a^4}, \quad (105)$$

where

$$\Delta = 36c_1 H^2 a^4 + 3c_1 R a^4 + 30c_1 \kappa a^2 + c_2 R a^2 + 18c_2 \kappa. \quad (106)$$

The Friedmann equation gives us the expression of f in terms of $R(a)$, $H(a)$ and a . Eq. (44), which can be rewritten here as

$$\frac{f'(a)}{R'(a)} = \frac{3a(c_1 a^2 + c_2) f(a) - \bar{c}(a^2 R(a) + 6\kappa)}{2a(c_2 R(a) - 6c_1 \kappa)} + \frac{(c_1 a^2 + c_2) \rho_{r0}}{2a^4(c_2 R(a) - 6c_1 \kappa)}, \quad (107)$$

giving a dynamics for f , defines a second order differential equation for H , given by

$$H'' = [(c_1 a^2 + c_2) H \Gamma]^{-1} H'^2 (12c_1^2 \bar{c} \kappa a^7 + 9c_1^3 \rho_{m0} a^7 + 54c_1^2 \mu^3 H a^7 + 12c_1^3 \rho_{r0} a^6 + 24c_1 c_2 \bar{c} \kappa a^5 + 21c_1^2 c_2 \rho_{m0} a^5 + 36c_1 c_2 \mu^3 H a^5 + 28c_1^2 c_2 \rho_{r0} a^4 + 12c_2^2 \bar{c} \kappa a^3 + 15c_1 c_2^2 \rho_{m0} a^3 + 6c_2^2 \mu^3 H a^3 + 20c_1 c_2^2 \rho_{r0} a^2 + 3c_2^3 \rho_{m0} a + 4c_2^3 \rho_{r0}) - [a(c_1 a^2 + c_2) H \Gamma]^{-1} H' (54c_1^2 \bar{c} H^3 a^9 + 108c_1 c_2 \bar{c} H^3 a^7 - 270c_1^2 \mu^3 H^2 a^7 - 60c_1^2 \bar{c} \kappa H a^7 - 45c_1^3 \rho_{m0} H a^7 - 72c_1^3 \rho_{r0} H a^6 + 36c_1^2 \kappa \mu^3 a^5 + 54c_2^2 \bar{c} H^3 a^5 - 324c_1 c_2 \mu^3 H^2 a^5 - 120c_1 c_2 \bar{c} \kappa H a^5 - 111c_1^2 c_2 \rho_{m0} H a^5 - 176c_1^2 c_2 \rho_{r0} H a^4 + 24c_1 c_2 \kappa \mu^3 a^3 - 78c_2^2 \mu^3 H^2 a^3 - 60c_2^2 \bar{c} \kappa H a^3 - 87c_1 c_2^2 \rho_{m0} H a^3 - 136c_1 c_2^2 \rho_{r0} H a^2 - 12c_2^2 \kappa \mu^3 a - 21c_2^3 \rho_{m0} H a - 32c_2^3 \rho_{r0} H) - [a^4 (c_1 a^2 + c_2) H^2 \Gamma]^{-1} 4(-18c_1 c_2 \mu^3 H^4 a^7 - 3c_1^2 c_2 \rho_{m0} H^3 a^7 + 18c_1^2 \kappa \mu^3 H^2 a^7 + 6c_1^2 \bar{c} \kappa^2 H a^7 + 3c_1^3 \kappa \rho_{m0} H a^7 - 6c_1^2 c_2 \rho_{r0} H^3 a^6 + 6c_1^3 \kappa \rho_{r0} H a^6 - 6c_2^2 \mu^3 H^4 a^5 - 6c_1^2 \kappa^2 \mu^3 a^5 - 6c_1 c_2^2 \rho_{m0} H^3 a^5 + 12c_1 c_2 \kappa \mu^3 H^2 a^5 + 12c_1 c_2 \bar{c} \kappa^2 H a^5 + 6c_1^2 c_2 \kappa \rho_{m0} H a^5 - 12c_1 c_2^2 \rho_{r0} H^3 a^4 + 14c_1^2 c_2 \kappa \rho_{r0} H a^4 - 12c_1 c_2 \kappa^2 \mu^3 a^3 - 3c_2^3 \rho_{m0} H^3 a^3 - 6c_2^2 \kappa \mu^3 H^2 a^3 + 6c_2^2 \bar{c} \kappa^2 H a^3 + 3c_1 c_2^2 \kappa \rho_{m0} H a^3 - 6c_2^3 \rho_{r0} H^3 a^2 + 10c_1 c_2^2 \kappa \rho_{r0} H a^2 - 6c_2^2 \kappa^2 \mu^3 a + 2c_2^3 \kappa \rho_{r0} H), \quad (108)$$

where

$$\Gamma = 18c_1 \bar{c} H^2 a^7 + 18c_2 \bar{c} H^2 a^5 - 12c_1 \bar{c} \kappa a^5 - 9c_1^2 \rho_{m0} a^5 - 54c_1 \mu^3 H a^5 - 12c_1^2 \rho_{r0} a^4 - 12c_2 \bar{c} \kappa a^3 - 12c_1 c_2 \rho_{m0} a^3 - 18c_2 \mu^3 H a^3 - 16c_1 c_2 \rho_{r0} a^2 - 3c_2^2 \rho_{m0} a - 4c_2^2 \rho_{r0}. \quad (109)$$

It is evident that a more detailed (numerical) study, pursued elsewhere, of this differential equation is necessary in order to study the dynamics of these solutions.

D. Non-Noether solutions

In general it is not possible to find a solution of the Friedmann equations which is also a Noether symmetry since, in principle, such symmetries do not exist for any $f(R)$ theory. In general, a solutions of the cosmological equations is not a solution compatible with the condition $L_{\mathbf{X}}\mathcal{L} = 0$. This is a peculiar situation which holds only if conserved quantities (Noether's charges) are intrinsically present in the structure of the theory (in our case, the form of $f(R)$). For example, imposing a power law solution, $a \propto t^p$, defines a function of $R = R(a)$, which can be put in the Noether symmetry equations, in order to find $f = f(R(a))$. Finally one can substitute the expressions for $f(a)$, $R(a)$, and H in the Friedmann equations. In doing this, it is easy to show that, for $k = 0$, there are no simple power-law solutions compatible with a Noether charge.

The method discussed above allows to discriminate theories which admit or not cosmological solutions compatible with a Noether charge.

It is also clear that power-law solutions do exist in general for $f(R)$ models, but they can be found using different methods [24]. Assuming, in general, a power-law $H(a)$, one finds R as a function of a , and then, in principle, $f = f(R(a))$. It is therefore possible to write the Einstein equation as a second order differential equation for f as a function of a , whereas all other quantities (H and R) are given functions of a . The same argument holds for the redshift z [21].

For example, let us rewrite the Friedmann equation (8) as

$$f - 6 f_{RR} \dot{R} H - 6 f_R H^2 - f_R \left(R + \frac{6k}{a^2} \right) = \frac{\rho_{m0}}{a^3} + \frac{\rho_{r0}}{a^4}, \quad (110)$$

and let us consider $H = \bar{H}(a)$ and $R = \bar{R}(a)$ as given functions of a , being, as above,

$$\bar{R} = -12 \bar{H}^2 - 6 a \bar{H} \bar{H}' - 6 \frac{k}{a^2}. \quad (111)$$

The Friedmann equation can be written as

$$f'' + \left[\frac{1}{a} - \frac{\bar{R}''}{\bar{R}'} + \frac{1}{6a \bar{H}^2} \left(\bar{R} + \frac{6k}{a^2} \right) \right] f' - \frac{\bar{R}'}{6a \bar{H}^2} f = -\frac{\rho_{m0} a + \rho_{r0}}{6 a^5 \bar{H}^2} \bar{R}'. \quad (112)$$

This is a second order linear equation in f , whose general solutions depends on two parameters, f_0 and f'_0 . Specifically, being the equation linear, the general solution is the linear combination of two solutions of the homogeneous ODE plus a particular solution. It is then clear that more than one $f(R)$ model can have the same behavior for $H(a)$, i.e. more theories share the same cosmological evolution. This situation is due to the fact that one has a fourth-order gravity theory. The singular points of this differential equation are those for which either \bar{H} or $d\bar{R}/da$ vanishes.

Starting from these considerations, interesting classes of solutions can be found out.

1. Radiation solutions

Let us seek for all the $f(R)$ models which have the particular solution $a = \sqrt{t/t_0}$, which means

$$\bar{H} = \frac{1}{2 t_0 a^2} = \frac{H_0}{a^2}, \quad \text{so that} \quad \bar{R} = -\frac{6k}{a^2}, \quad (113)$$

where $H_0 \equiv (2 t_0)^{-1}$. We have three interesting cases.

1. For $k = 0$, we have $R = 0$, leading to the Friedmann equation

$$f(0) - 6 f_R(0) \bar{H}^2 = \frac{\rho_{m0}}{a^3} + \frac{\rho_{r0}}{a^4}, \quad (114)$$

which, if $\rho_{m0} \neq 0$, cannot be solved for $\bar{H} \sim a^{-2}$ since $f(0)$ and $f'(0)$ cannot be functions of a , but only constants. If $\rho_{m0} = 0$, standard GR is of course recovered.

2. For the case $k = -1$ we have the following differential equation for f ,

$$f'' + \frac{4}{a} f' + \frac{2\kappa}{H_0^2} f = \frac{2\kappa(\rho_{r0} + a\rho_{m0})}{H_0^2 a^4}, \quad (115)$$

whose general solution can be written as

$$\begin{aligned}
 R &= -\frac{6\kappa}{a^2} \\
 f &= \frac{\sqrt{\frac{a\sqrt{-\kappa}}{H_0}} d_2 \cos\left(\frac{a\sqrt{-2\kappa}}{H_0}\right) H_0^2}{\sqrt[4]{2} a^{7/2} \kappa \sqrt{\pi}} - \frac{\sqrt{\frac{a\sqrt{-\kappa}}{H_0}} d_1 \sin\left(\frac{a\sqrt{-2\kappa}}{H_0}\right) H_0^2}{\sqrt[4]{2} a^{7/2} \kappa \sqrt{\pi}} - \frac{\sqrt[4]{2} \sqrt{\frac{a\sqrt{-\kappa}}{H_0}} d_1 \cos\left(\frac{a\sqrt{-2\kappa}}{H_0}\right) H_0}{a^{5/2} \sqrt{-\kappa} \sqrt{\pi}} \\
 &\quad - \frac{\sqrt[4]{2} \sqrt{\frac{a\sqrt{-\kappa}}{H_0}} d_2 \sin\left(\frac{a\sqrt{-2\kappa}}{H_0}\right) H_0}{a^{5/2} \sqrt{-\kappa} \sqrt{\pi}} + \frac{\rho_{m0}}{a^3} + \frac{\rho_{r0} \sqrt{-\kappa} \text{Ci}\left(\frac{\sqrt{2} a \sqrt{-\kappa}}{H_0}\right) \sin\left(\frac{a\sqrt{-2\kappa}}{H_0}\right)}{\sqrt{2} a^3 H_0} \\
 &\quad - \frac{\rho_{r0} \sqrt{-\kappa} \cos\left(\frac{a\sqrt{-2\kappa}}{H_0}\right) \text{Si}\left(\frac{a\sqrt{-2\kappa}}{H_0}\right)}{\sqrt{2} a^3 H_0} + \frac{\rho_{r0} \kappa \cos\left(\frac{a\sqrt{-2\kappa}}{H_0}\right) \text{Ci}\left(\frac{a\sqrt{-2\kappa}}{H_0}\right)}{a^2 H_0^2} \\
 &\quad + \frac{\rho_{r0} \kappa \sin\left(\frac{a\sqrt{-2\kappa}}{H_0}\right) \text{Si}\left(\frac{a\sqrt{-2\kappa}}{H_0}\right)}{a^2 H_0^2},
 \end{aligned} \tag{116}$$

where the SinIntegral and CosIntegral functions, Si and Ci respectively, are defined as

$$\text{Si}(x) = \int_0^x \frac{\sin(t)}{t} dt \quad \text{Ci}(x) = -\int_x^\infty \frac{\cos(t)}{t} dt. \tag{118}$$

The integration constants $d_{1,2}$ have dimensions $[d_1] = [d_2] = M^4$.

3. Along the same lines, the case $k = 1$ has the following solution

$$\begin{aligned}
 R &= -\frac{6\kappa}{a^2} \\
 f &= \frac{\sqrt{\frac{a\sqrt{\kappa}}{H_0}} d_1 \cosh\left(\frac{\sqrt{2\kappa} a}{H_0}\right) H_0^2}{\sqrt[4]{2} a^{7/2} \kappa \sqrt{\pi}} + \frac{\sqrt{\frac{a\sqrt{\kappa}}{H_0}} d_1 \sinh\left(\frac{\sqrt{2\kappa} a}{H_0}\right) H_0^2}{\sqrt[4]{2} a^{7/2} \kappa \sqrt{\pi}} - \frac{\sqrt[4]{2} \sqrt{\frac{a\sqrt{\kappa}}{H_0}} d_1 \cosh\left(\frac{\sqrt{2\kappa} a}{H_0}\right) H_0}{a^{5/2} \sqrt{\pi \kappa}} \\
 &\quad - \frac{\sqrt[4]{2} \sqrt{\frac{a\sqrt{\kappa}}{H_0}} d_1 \sinh\left(\frac{\sqrt{2\kappa} a}{H_0}\right) H_0}{a^{5/2} \sqrt{\pi \kappa}} + \frac{\rho_{m0}}{a^3} - \frac{\rho_{r0} \sqrt{\kappa} \text{Chi}\left(\frac{\sqrt{2\kappa} a}{H_0}\right) \sinh\left(\frac{\sqrt{2\kappa} a}{H_0}\right)}{\sqrt{2} a^3 H_0} \\
 &\quad + \frac{\rho_{r0} \sqrt{\kappa} \cosh\left(\frac{\sqrt{2\kappa} a}{H_0}\right) \text{Shi}\left(\frac{\sqrt{2\kappa} a}{H_0}\right)}{\sqrt{2} a^3 H_0} - \frac{\rho_{r0} \kappa \cosh\left(\frac{\sqrt{2\kappa} a}{H_0}\right) \text{Chi}\left(\frac{\sqrt{2\kappa} a}{H_0}\right)}{a^2 H_0^2} \\
 &\quad + \frac{\rho_{r0} \kappa \sinh\left(\frac{\sqrt{2\kappa} a}{H_0}\right) \text{Shi}\left(\frac{\sqrt{2\kappa} a}{H_0}\right)}{a^2 H_0^2},
 \end{aligned} \tag{119}$$

where the hyperbolic SinIntegral and CosIntegral, Shi and Chi respectively, are defined as

$$\text{Shi}(x) = \int_0^x \frac{\sinh(t)}{t} dt \quad \text{Chi}(x) = \gamma_{E,M} + \ln(x) + \int_0^x \frac{\cosh(t) - 1}{t} dt, \tag{121}$$

and $\gamma_{E,M} \approx 0.577$ is the Euler-Mascheroni constant. Both d_1 and d_2 are integration constants which dimensions M^4 .

2. Matter solutions

In this case, we search for $f(R)$ models which have a dust-matter behavior, that is $a = (t/t_0)^{2/3}$,

$$\bar{H} = \frac{2}{3 t_0 a^{3/2}} = \frac{H_0}{a^{3/2}}, \quad \text{and} \quad \bar{R} = -\frac{2(2/t_0^2 + 9\kappa a)}{3 a^3}, \tag{122}$$

where $H_0 \equiv 2/(3 t_0)$. For the case $k = 0$, we find the explicit analytic solution

$$R = -\frac{4}{3 t_0^2 a^3}, \tag{123}$$

$$f(a) = a^{-(7+\sqrt{73})/4} \left(d_1 a^{\sqrt{73}/2} + d_2 \right) + \frac{\rho_{m0} a - 6\rho_{r0}}{2 a^4}. \tag{124}$$

This is a 2-parameters family of solutions, depending on the two integration constants $d_{1,2}$ both with dimensions M^4 . The Einstein-Hilbert case $f(R) = R$ belongs to this family, when d_1 , d_2 , and ρ_{r0} all vanish.

3. Exponential solutions

In this case, we look for the behavior

$$\bar{H} = H_0 = \text{constant}, \quad \text{which is} \quad \bar{R} = -12 H_0^2 - \frac{6k}{a^2}. \quad (125)$$

As above, we have three cases depending on k .

1. $k = 0$. Both H and R are constants, and $R = R_0 \equiv -12 H_0^2$. The Friedmann equation is

$$f(R_0) - \frac{1}{2} f_R(R_0) R_0 = \frac{\rho_{m0}}{a^3} + \frac{\rho_{r0}}{a^4}, \quad (126)$$

and it has solutions only for $\rho_{m0} = \rho_{r0} = 0$ being R_0 a constant (see also [53]).

2. $k = 1$. In this case, H is still a constant but R is not. One finds

$$R = -12 H_0^2 - \frac{6\kappa}{a^2} \quad (127)$$

$$\begin{aligned} f = & d_1 \cosh\left(\frac{\sqrt{2\kappa}}{H_0 a}\right) + d_2 \sinh\left(\frac{\sqrt{2\kappa}}{H_0 a}\right) \\ & + \frac{6\rho_{r0} H_0^4}{\kappa^2} + \frac{3\rho_{m0} H_0^2}{a\kappa} + \frac{6\rho_{r0} H_0^2}{a^2\kappa} + \frac{\rho_{r0}}{a^4} + \frac{\rho_{m0}}{a^3}. \end{aligned} \quad (128)$$

3. $k = -1$. The solution is

$$R = -12 H_0^2 - \frac{6\kappa}{a^2}, \quad (129)$$

$$\begin{aligned} f = & d_1 \cos\left(\frac{\sqrt{-2\kappa}}{H_0 a}\right) + d_2 \sin\left(\frac{\sqrt{-2\kappa}}{H_0 a}\right) \\ & + \frac{6\rho_{r0} H_0^4}{\kappa^2} + \frac{3\rho_{m0} H_0^2}{a\kappa} + \frac{6\rho_{r0} H_0^2}{a^2\kappa} + \frac{\rho_{r0}}{a^4} + \frac{\rho_{m0}}{a^3}. \end{aligned} \quad (130)$$

4. Λ CDM solutions

Let us now look for $f(R)$ models which are compatible with the Λ CDM being solutions of Friedmann equations. This analysis could be extremely important to compare the $f(R)$ approach with observations (see also [47]). One defines

$$\bar{H}^2 = H_0^2 \left[\frac{\Omega_{m0}}{a^3} + \frac{\Omega_{r0}}{a^4} + 1 - \Omega_{m0} - \Omega_{r0} \right]. \quad (131)$$

The differential equation to solve is therefore the following

$$\begin{aligned} f'' + & \left[\frac{6\Omega_{m0}H_0^2}{3\Omega_{m0}H_0^2 + 4ak} - \frac{4(\Omega_{m0} + \Omega_{r0} - 1)a^4 - 7\Omega_{m0}a - 8\Omega_{r0}}{-(\Omega_{m0} + \Omega_{r0} - 1)a^4 + \Omega_{m0}a + \Omega_{r0}} \right] \frac{f'}{2a} \\ & - \frac{3\Omega_{m0}H_0^2 + 4ak}{2a [-(\Omega_{m0} + \Omega_{r0} - 1)a^4 + \Omega_{m0}a + \Omega_{r0}] H_0^2} f \\ = & - \frac{(3\Omega_{m0}H_0^2 + 4ak)(\rho_{r0} + a\rho_{m0})}{2a^5 [-(\Omega_{m0} + \Omega_{r0} - 1)a^4 + \Omega_{m0}a + \Omega_{r0}] H_0^2}, \end{aligned} \quad (132)$$

The general integral can be numerically achieved by giving suitable initial conditions for f_0 , f'_0 . This analysis will be pursued in a forthcoming paper.

VI. DISCUSSION AND CONCLUSIONS

In this paper, we have discussed a general method to find out exact/analytical cosmological solutions in $f(R)$ gravity. The approach is based on the search for Noether symmetries which allow to reduce the dynamics and, in principle, to solve more easily the equations of motion. Besides, due to the fact that such symmetries are always related to conserved quantities, such a method can be seen as a physically motivated criterion.

The main point is that the existence of the symmetry allows to fix the form of $f(R)$ models assumed in a point-like cosmological action where the FLRW metric is imposed. It is worth noticing that, starting from a point-like FLRW Lagrangian, and then deriving the Euler-Lagrange equations of motion, leads exactly to the same equations obtained by imposing the FLRW metric in the Einstein field equations. This circumstance allows to search “directly” the Noether symmetries in the point-like Lagrangian and then to plug the related conserved quantities into the equations of motion. As a result *i*) the form of the $f(R)$ is fixed directly by the symmetry existence conditions and *ii*) the dynamical system is reduced since some of its variables (at least one) is cyclic.

The method is useful not only in a cosmological context but it works, in principle, every time a canonical, point-like Lagrangian is achieved (in [54], it has been used to find out spherically symmetric solutions in $f(R)$ gravity).

In this paper, we have considered a generic $f(R)$ theory where standard fluid matter (dust and radiation) is present. The Noether conditions for symmetry select forms of $f(R)$ depending on a set of cosmological parameters such as $\{\rho_{r0}, \rho_{m0}, k, H_0\}$ and the effective gravitational coupling. Such a dependence can be easily translated into the more suitable set of observational parameters $\{\Omega_{r0}, \Omega_{m0}, \Omega_k, H_0\}$ and then matched with data. This situation has a twofold relevance: from one side, it could contribute to remove the well known problem of degeneracy (several dark energy models fit the same data and, essentially, reproduce the Λ CDM model); from the other side, being the search for Noether symmetries a relevant approach to find out conserved quantities in physics, this could be an interesting method to select models motivated at a fundamental level. It is worth noticing that the Noether constant of motion, which we have found, has the dimensions of a mass and is directly related to the various sources present into dynamics. In some sense, the Noether constant “determines” the bulk of the various sources as ρ_{m0} , ρ_{r0} and the effective ρ_Λ and then could greatly contribute to solve the dark energy and dark matter puzzles. In a forthcoming paper, we will directly compare the solutions which we have presented here with observational data.

The “non-Noether solutions” deserve a final remark. In this case, we do not ask for a Noether symmetry but, finding these solutions, can be related to the previous general method. We have shown that the standard cosmological behaviors of the usual Einstein-Friedmann cosmology can be achieved also in generic $f(R)$ models, assuming that the cosmological quantities H and R depend on the scale factor a . As result, we find out general $f(R(a))$ where the standard solutions of the linear $f(R) = R$ case are easily recovered.

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Appendix A: Solutions and link with scalar-tensor theories

We will explicitly show, as an example, that equation (102) is indeed a Noether solution (with $k = 0$, flat space, and $\mu_0 = 0$, zero Noether charge). First, from $H(a)$, given by

$$\begin{aligned}
 H^2 = & -\frac{4d_1d_2(-c_3)^{9/2}}{a^4} + \frac{24d_1d_2(-c_3)^{7/2}}{a^2} + \frac{\rho_{0m}d_2(-c_3)^{5/2}}{a^4} - 36d_1d_2(-c_3)^{5/2} \\
 & + \frac{2\sqrt{3}\rho_{r0}\text{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right)d_2c_3^2}{a^4} + \frac{10\rho_{r0}d_2(-c_3)^{3/2}}{a^3} + \frac{12\sqrt{3}\rho_{r0}\text{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right)d_2c_3}{a^2} \\
 & - \frac{18\rho_{r0}d_2\sqrt{-c_3}}{a} + 18\sqrt{3}\rho_{r0}\text{arctanh}\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right)d_2,
 \end{aligned} \tag{A1}$$

we can calculate the expression for $R(a)$ as follows

$$\begin{aligned}
R &= -12 H^2 - 6 a H H' \\
&= -\frac{144 d_1 d_2 (-c_3)^{7/2}}{a^2} + 432 d_1 d_2 (-c_3)^{5/2} - \frac{48 d_2 \rho_{r0} (-c_3)^{3/2}}{a^3} - \frac{72 \sqrt{3} d_2 \rho_{r0} \operatorname{arctanh}\left(\frac{\sqrt{3} a}{\sqrt{-c_3}}\right) c_3}{a^2} \\
&\quad + \frac{216 d_2 \rho_{r0} \sqrt{-c_3}}{a} - 216 \sqrt{3} d_2 \rho_{r0} \operatorname{arctanh}\left(\frac{\sqrt{3} a}{\sqrt{-c_3}}\right). \tag{A2}
\end{aligned}$$

Since we know both H and R , now, by using Eq. (88), we can find $f(a)$ as follows

$$f = -\frac{8 d_1 c_3^2}{a^3} - \frac{24 d_1 c_3}{a} - \frac{3 \rho_{r0}}{a^4} + \frac{4 \sqrt{3} \rho_{r0} \operatorname{arctanh}\left(\frac{\sqrt{3} a}{\sqrt{-c_3}}\right)}{a^3 \sqrt{-c_3}} - \frac{12 \rho_{r0}}{a^2 c_3} - \frac{12 \sqrt{3} \rho_{r0} \operatorname{arctanh}\left(\frac{\sqrt{3} a}{\sqrt{-c_3}}\right)}{a (-c_3)^{3/2}}. \tag{A3}$$

These expressions for f, R, H fulfill equation (9). The system has also a constant of motion $\mu_0 = 0$ given by equation (48), as the Lagrangian possesses a Noether symmetry.

We will discuss how to link this solution (extending this procedure to the other solutions is straightforward) to the scalar-tensor picture, by finding the potential for the scalar non-minimally coupled with gravity. In fact, starting from the action

$$S = \int d^4 x \sqrt{-g} f(R) + S_m, \tag{A4}$$

one can rewrite it (at least at the classical level) in the following form

$$S = \int d^4 x \sqrt{-g} [f_\varphi R - V(\varphi)] + S_m, \tag{A5}$$

where $V = \varphi f_\varphi - f(\varphi)$, and $f_\varphi = \partial f / \partial \varphi$. The classical equation of motion for φ leads to $\varphi = R$. One can make a field redefinition to bring the action in the form

$$S = \int d^4 x \sqrt{-g} [-\chi R - V(\chi)] + S_m, \tag{A6}$$

where $\chi = -f_\varphi$.

In this case we can use our solutions in order to find $V(\chi)$, the only unknown in the theory. One can do it as follows

$$\chi = -f_\varphi = -f_R = -\frac{f'}{R'} \tag{A7}$$

$$V = \varphi f_\varphi - f = R f_R - f = R \frac{f'}{R'} - R, \tag{A8}$$

where these relations are correct on shell, i.e. for the solutions of the equations of motion. Using equations (A1),

(A2), and (A3), one can write down explicitly the potential, at least for this case, as follows

$$\begin{aligned}
V(\chi) = & 3456d_1d_2^3\chi^3(-c_3)^{13/2} - 10368d_2^4\rho_{r0}\chi^4c_3^6 \\
& - 1728\sqrt{3}d_2^3\rho_{r0}\chi^3\operatorname{arctanh}\left[\frac{\sqrt{3}\left(6d_2\chi(-c_3)^{5/2} + \sqrt{-36d_2^2\chi^2c_3^5 - c_3}\right)}{\sqrt{-c_3}}\right]c_3^4 \\
& + 1728d_2^3\rho_{r0}\chi^3\sqrt{-36d_2^2\chi^2c_3^5 - c_3}(-c_3)^{7/2} - 288d_1d_2\chi(-c_3)^{5/2} + 432d_2^2\rho_{r0}\chi^2c_3^2 \\
& + 288\sqrt{3}d_2^2\rho_{r0}\chi^2\sqrt{-36d_2^2\chi^2c_3^5 - c_3}\operatorname{arctanh}\left[\frac{\sqrt{3}\left(6d_2\chi(-c_3)^{5/2} + \sqrt{-36d_2^2\chi^2c_3^5 - c_3}\right)}{\sqrt{-c_3}}\right](-c_3)^{3/2} \\
& + 144\sqrt{3}d_2\rho_{r0}\chi\operatorname{arctanh}\left[\frac{\sqrt{3}\left(6d_2\chi(-c_3)^{5/2} + \sqrt{-36d_2^2\chi^2c_3^5 - c_3}\right)}{\sqrt{-c_3}}\right] - 16d_1\sqrt{-36d_2^2\chi^2c_3^5 - c_3} \\
& - \frac{96d_2\rho_{r0}\chi\sqrt{-36d_2^2\chi^2c_3^5 - c_3}}{\sqrt{-c_3}} - \frac{9\rho_{r0}}{c_3^2} - 576d_1d_2^2\chi^2\sqrt{-36d_2^2\chi^2c_3^5 - c_3}c_3^4 \\
& + 8\sqrt{3}\rho_{r0}(-c_3)^{-5/2}\sqrt{-36d_2^2\chi^2c_3^5 - c_3}\operatorname{arctanh}\left[\frac{\sqrt{3}\left(6d_2\chi(-c_3)^{5/2} + \sqrt{-36d_2^2\chi^2c_3^5 - c_3}\right)}{\sqrt{-c_3}}\right]. \quad (A9)
\end{aligned}$$

In order to study the evolution of the background, whether or not it leads to a viable dynamics for the universe, it is already sufficient to check if the Hubble parameter given by (A1) can fit the data, from Big Bang Nucleosynthesis up to Dark Energy domination.

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